# Increased Frequency of Defectors Using Evolutionary Spatial Modeling of the Snowdrift game

*Robert Yates*

**Abstract**

Evolutionary game theory models a population of agents playing games. Over time an evolutionarily stable strategy (ESS) that is resistant to invasion by other strategies can emerge. Prisoner's dilemma and snowdrift are two common games used in game theory, with the former being quite popular. Iterated repetitions of both games promote cooperation. [5] We can use evolutionary game theory to model a population of agents playing both games. Typically, we model agents over well-mixed populations. Alternatively, Hauert et al. [3] investigate using a spatial structure to model a population, i.e. individual agents play games only against their neighbors in the structure. Spatial structure increases cooperation for all variations of prisoner's dilemma and when the cost is much smaller than the benefit in the snowdrift game. However, spatial structure can decrease cooperation and increase defection when the cost is slightly smaller to moderately smaller than the benefit in the snowdrift game.

## 1 Evolutionary Game Theory

According to Maynard Smith [9] in his seminal work 'Evolution and the Theory of Games", we can express evolutionary game theory as a way of thinking about how physical characteristics (phenotype) of individuals develop across a population. First introduced by Maynard Smith and Price [8], evolutionary game theory enables us to model how the behavior of animals populations relates to how frequent a trait appears in a given population. Game theory literature uses the concept of many agents which act according to a rational self-interest. Evolutionary game theory is concerned with fitness, replicator dynamics and evolutionary stable strategies. Previously literature on game theory in biology by R.C. Lewontin [6] modeled animals as playing games against nature. However, Maynard Smith modeled the population of a single species where individuals of that species play games against one another. Evolutionary game theory, as Maynard Smith describes it, originated out of work by R.A. Fisher. In 1930, Fisher [4][1] researched why there was always a roughly equal number of males and females in many species of mammals. Fisher used the number of grandchildren of an individual to represent the Darwinian fitness strength to explain the equal sex ratio. If there are more males in a population, females have higher individual fitness; and if there are more females, males have higher individual fitness. Over time, the number of both males and females in a population even out to a roughly equal number of each.

## 2 Replicator Dynamics and ESS

As discussed in Ch. 7.7, p.224 of the textbook for this course, "Multiagent Systems: Algorithmic, Game-Theoretic, and Logical Foundations" [10], evolutionary game theory models

the learning of a population of agents instead of individual agents. We will discuss the replicator dynamic and evolutionarily stable strategies (ESS).
The replicator dynamic of a single population is how the agents interact over time. The agents within a population all play pure strategies. The only difference between agents is which pure strategy they choose. After every time update, the agents reproduce according to their fitness or total payoff. We analyze the frequency of agents playing each pure strategy and see if these frequencies converge to fixed value as time approaches infinity.
Economists rely on the process of rational decision making, while evolutionary biologists think of natural selection as the decision maker. Animal adaptations reflect the process of evolution rather than mental processes involved in rational decision making [2]. Evolutionarily stable strategy is an additional way to model a stable solution of strategies coexisting in a population of agents (animals). An evolutionarily stable strategy is a mixed strategy that is resistant to invasion by new strategies. A strategy is an ESS if the payoff of just the ESS strategy is higher than the payoff of the ESS strategy and some new invader strategy. As formalized by Maynard Smith [9] [2], if all members of a population are playing an evolutionarily stable strategy (ESS), then another strategy could not successfully invade the population.

# 3 Games

| | H | D |
|---|---|---|
| H | (V-C)/2 | V |
| D | 0 | V/2 |

Table 1: Hawk-Dove game

| | C | D |
|---|---|---|
| C | R | S |
| D | T | P |

Table 2: Prisoner's dilemma

| | C | D |
|---|---|---|
| C | b-(c/2) | b-c |
| D | b | 0 |

Table 3: Snowdrift game

The following are three types of games commonly used in game theory. All games are symmetric. The hawk-dove game was introduced by Maynard Smith for evolutionary game theory and discussed in chapter 7 of “Multiagent Systems: Algorithmic, Game-Theoretic, and Logical Foundations” [10]. Prisoner's dilemma is a commonly used in game theory literature. In the snowdrift game, two agents work to obtain a common benefit (b) at some shared cost c to the cooperating agents.
The snowdrift game can be explained as follows. Suppose there is snowdrift on the highway late at night. Two drivers on this road are returning to their respective homes and driving in opposite directions. One driver encounters the snowdrift on the north side, the other on the south side. If both drivers shovel out the snow (both cooperate, C), then they both get the benefit of going home (b) minus the split cost of shoveling (c/2). If neither driver shovels the snow (both defect, D), then neither gets to go home so the payoff for each is 0. If only one driver shovels while the other driver does nothing, the driver that shovels (cooperate C) gets the benefit of going home (b) minus the cost of shoveling (c), but the driver that does nothing (defect D) also gets the benefit of going home (b) without the cost.

The payoffs for prisoner's dilemma are $T > R > P > S$. Note that both the hawk-dove game and the snowdrift game are instances of the prisoner's dilemma if we change the payoffs such that $V > C$ in the hawk-dove game and $c > b$ in the snowdrift game. Hence we make sure that $C > H$ in the hawk-dove game and $b > c$ in the snowdrift game so these games are not instances of the prisoner's dilemma.
In prisoner's dilemma, $T > R > P > S$, so the dominant strategy is to always defect. So (always defect, always defect) is a pure strategy Nash equilibrium. In the snowdrift game, the S and P payoffs are flipped around: $T > R > S > P$. This means the best response to cooperation is defection and the best response to d is defection is cooperation.
The hawk-dove game can be modeled using the snowdrift game. In the snowdrift game, we can set the payoffs $b = \frac{\beta+\gamma}{2}$ and $c = \beta$ such that $\gamma > \beta$. Then the game differs from the hawk-dove game by only a constant factor of $\frac{\beta-\gamma}{2}$ so both games become essentially equivalent.
Recall from homework 3, the evolutionarily stable strategy of the hawk-dove game is for each player to play H with probability V/C and each player to play D with probability 1-(V/C). Using the same logic, the evolutionarily stable strategy of the snowdrift game is for each player to play C with probability $\frac{2b-2c}{2b-c}$ and each player to play D with probability $\frac{c}{2b-c}$. (See appendix for computation of snowdrift evolutionarily stable strategy.)

# 4 Prior Work

In non-iterated snowdrift, the best response to cooperating is defecting and the best response to defecting is cooperating. In non-iterated prisoner's dilemma, the dominant strategy is to always defect. However, this changes for iterated games.
Prior research by Kümmerli et al. [5] compared human players' strategies on iterated snowdrift and iterated prisoner's dilemma. They showed that in iterated snowdrift with a low risk of playing against a defector, that human players were likely to cooperate. (With a high risk of playing against a defector, humans were less likely to cooperate.) Overall, there was higher cooperation in iterated snowdrift than iterated prisoner's dilemma. In iterated prisoner's dilemma, playing tit-for-tat (cooperate by default, punish defection with defection) is an effective strategy resulting in a large expected payoff over many games.
Both prisoner's dilemma and snowdrift have been modeled in evolutionary game theory with a population of agents. In prisoner's dilemma, defection is an evolutionarily stable strategy. The evolutionarily stable strategy of the snowdrift game is calculated in the appendix. Prior research has shown that spatial structure promotes cooperation in the evolutionary prisoner's dilemma. We will investigate spatial structure in evolutionary snowdrift. [3]
In the snowdrift game, we set the independent variable $r = \frac{c}{2b-c}$, which we call the cost-benefit ratio. $r$ also is the frequency of defectors in a population in a well-mixed population of agents. When the cost $c$ is almost as large as the benefit $b$, then there is a high cost-benefit ratio, $r$. When the cost $c$ is much smaller than the benefit $b$, then there is a low cost-benefit ratio, $r$.

# 5 Spatial Structure Model

Hauert et al. [3] model a population of agents over time in a spatial structure. In particular, individuals in a population reside at points in a regular lattice. Let individuals have two strategies: either always cooperate or always defect. Over time, we repeatedly perform synchronous or asynchronous updates of points. When each point is updated, then the point's occupant and nearby neighbors compete in the snowdrift game. The individual that wins the point for a given update, will have his offspring occupy the site in the next round. The results of the spatial model are as follows: When $r$, the frequency of defectors, is low, the spatial model results in a higher frequency of cooperators and a lower frequency of defectors than in a well-mixed population without the spatial model. However, when $r$ is medium or high, the spatial model results in a higher frequency of defectors and a lower frequency of cooperators than in a well-mixed population without the spatial model. What is interesting is that prior research [3] has shown that the spatial model promotes cooperation. Remember, for a high $r = \frac{c}{2b-c}$, the cost is almost large as the benefit. When r is low, the cost is much less than the benefit. When the cost is greater than the benefit, the snowdrift game becomes an instance of the prisoner's dilemma. So the spatial model with a high $r$ should promote cooperation like in the prisoner's dilemma. But that's not the case here.
For all figures 1 and 2, we use a square lattice of a grid as our spatial model. Alternatively stated, the number of neighbors N = 4.

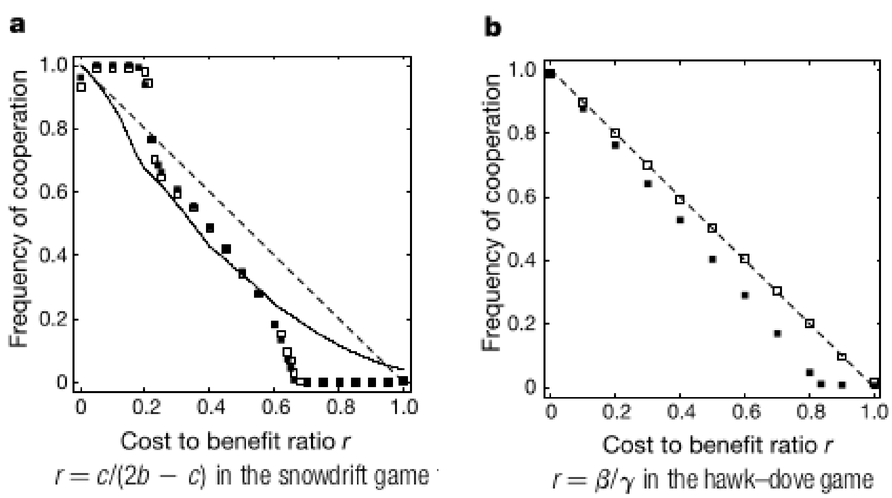


Figure 1: Near extinction of cooperators: a) snowdrift game b) hawk-dove game

In figure 1, we see the frequency of cooperators as a function of the independent cost-benefit ratio that we specify. Figure 1a models the snowdrift game where $r = \frac{c}{2b-c}$. Recall that the hawk-dove game can be modeled using the snowdrift game by setting the values of $b = \frac{\beta+\gamma}{2}$

and $c = \beta$ such that $\gamma > \beta$. Using these values in Figure 1b, we can set the cost-benefit ratio $r = \beta/\gamma$ to simulate the hawk-dove game. The dotted line in both 1 a and b represents the expected frequency of cooperations in a well-mixed population as we vary $r$, the cost-benefit ratio.

In Figure 1 a, we note there are two thresholds $r_1 = 0.2, r_2 = 0.65$. If we set the cost-benefit ratio $r < r_1$, the frequency of cooperations increases to 100% while the defectors vanish. If $r > r_2$, the frequency of cooperations decreases to 0% while the defectors dominate. A smaller N means a higher $r_1$ and a lower $r_2$. This occurs for both synchronous updates (modeled as the black squares squares) and asynchronous updates (modeled as the white squares).

In Figure 1 b, we simulate the hawk-dove game so $r_1$ is nonexistant, i.e. unless we set cost-benefit to 0, the frequency of cooperations never increases to 100% and defectors never vanish. However, if $r$ is greater than the cost-benefit ratio $r_2 = 0.8$, the frequency of cooperations decreases to 0% while the defectors dominate. Note in the hawk-dove game shown in Figure 1b, using asynchronous updates (modeled as the white squares), the cost-benefit ratio does not affect equilibrium strategy.

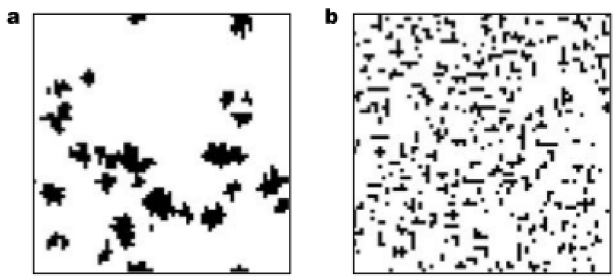


Figure 2: Near extinction of cooperators: a) prisoner's dilemma b) snowdrift game

In figure 2 a and b, we display a visualization of the spatial structure at the frequency at which cooperations disappear, where $r$ is almost above the $r_2$ threshold. Cooperators are shown in black and defectors in white. Figure 2a is the prisoner's dilemma game, 2b is the snowdrift game. Notice how in the prisoner's dilemma game cooperators clumb together into a few large clusters. This is due to the fact in prisoner's dilemma, the highest payoff for a cooperator agent is when his opponent is also a cooperator. In the snowdrift game cooperators are distanced themselves into a many smaller "t" like structures. This is due to the fact in snowdrift, the only payoff for a defector agent is when his opponent is a cooperator. Since the defector receives no playoff when he plays another defector, he is strongly incentivized to be near and play to a cooperator. Without loss of generality, the same results apply to neighborhood sizes $3 \leq N \leq 8$ as well.

# 6 Conclusion

Spatial structure typically promotes the evolution of cooperation. However in the snowdrift game with cost values slightly to moderately larger than benefit values, the spatial structure typically promotes the evolution of defection. In a real-life situation, the "closer" you are interpersonal-wise to another player, the more likely you are to defect with the costs are slightly less than, but no larger than, the benefits. This makes sense since the optimal strategy is to do the opposite of what you think the other agent will do. (If you think the other agent will cooperate, you will defect. If you think the other agent will defect, you will cooperate.)

Imagine a scenario with a large family of brothers and sisters. The siblings have different schedules, so they often eat no more than two at a time. After two siblings have just ate dinner, but before they have dessert, they have to decide who will wash the dishes from dinner. No clean dishes means no dessert. The benefit is the clean dishes and the dessert afterwards. The cost is the labor of washing the dishes. The siblings know each other very well. If one family member thinks the other will do the dishes, then he/she won't do them. But if the child thinks the other won't do the dishes, then he/she will do them since the dessert is quite tasty and doing dishes isn't much work. Our model says over many evenings in this household, the siblings will become lazy and skip doing the dishes. However, suppose we change the dessert to a new Playstation video game or another prize of each child's choosing. The benefit to doing dishes becomes much greater and hence we would see more cooperation when doing dishes. And suppose, instead of siblings we carried this experiment out in a university lab with two students who didn't know each other. Not knowing anything about the other student, each student may volunteer to do the dishes as they won't be certain of the other student's strategy.

In short, you are better off cooperating when playing the snowdrift game against strangers in a well-mixed population. If you only play people you know well and the cost is almost as large as the benefit, then you are better off defecting.

Hauert et al. [3] explain that despite it being advantageous for agents to play the strategy which is the opposite of their neighbors, it is always best to have cooperating neighbors. Cooperation is maintained in well-mixed populations precisely because it is better to do the opposite of your opponent. The clusters of cooperators that exist in spatial prisoner's dilemma don't exist in the spatial snowdrift game for this reason, hence the increasing rate of defectors. While prior research [3] showed only the positives of spatial structure (i.e. promoting cooperation), Hauert et al. have shown the downsides as well.

# 7 Final Thoughts and Opinion

As discussed by Kümmerli et al. [5], it can be argued that the snowdrift game more accurately depicts real-life situations the prisoner's dilemma. More often than not, there is a public good that can be exploitable by defectors who free-ride on cooperators, however the cooperators still receive greater benefit than the cost of their work. The snowdrift game represents the situation where there are direct benefits to all agents when at least one or both are cooperative, and the costs are shared between those who cooperated. Another example could be two students working on a group project or two researchers collaborating on a publication. It is best for everyone when both students/researchers contribute work, but if one person isn't contributing, the other can still complete the project at a large cost to himself. Hence the snowdrift game may be underrepresented in game theory, while the prisoner's dilemma may be overrepresented.

I feel there should be stronger collaboration between game theorists and evolutionary biologists in order to generate models of that more closely reflect zoological behavior. Hauert et al. [3] give an example of RNA phages that engage in prisoner's dilemma at first, but then the payoffs change to that of the snowdrift game with both cooperators and defectors in the stable steady state. Further research might allow us to understand differentness in classes of individuals as well. For example, Kümmerli et al. [5] showed in iterated prisoner's dilemma that women cooperate more often than men and receive a higher overall payoff. They did this by employing the strategy of tit-for-tat or "Pavlov" (repeat previous move if opponent cooperates, do opposite of previous move if opponent defects). However men were also more likely to cooperate when playing against women than other men. In addition, new research by Martin et al. [7] suggests that chimpanzees play strategies closer to equilibrium perfection that human subjects. Research by Hauert et al. [3] can be extended by modeling chimpanzees and humans differently, men and women differently and seeing how they affect the model. In summary, not all animals play the same games the same way.

# References


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# 8 Appendix: Evolutionary Game Theory Definitions

Pages 225 through 299 in the "Multiagent Systems: Algorithmic, Game-Theoretic, and Logical Foundations" textbook for this course [10] contain further discussion and definitions of evolutionary game theory topics such as replicator dynamics and evolutionarily stable strategy (ESS).

# 9 Appendix: Evolutionarily Stable Strategy of the Snowdrift game

| | C | D |
|---|---|---|
| C | b-(c/2) | b-c |
| D | b | 0 |

To differentiate snowdrift from the prisoner's dilemma, we assume the benefit is always greater than the cost $b > c$. Because of this, there will be both cooperators and defectors in a diverse population of agents. Over time, the frequency of cooperators becomes $\frac{2b-2c}{2b-c}$ and the frequency of defectors becomes $\frac{c}{2b-c}$ in a population. This is the evolutionarily stable strategy.
Utility for Player 2 playing C given Player 1 plays p,1-p =
Utility for Player 2 playing D given Player 1 plays p,1-p
$(b - \frac{c}{2})p + (b - c)(1 - p) = bp + 0(1 - p)$
$bp - \frac{c}{2}p + b - c - bp + cp = bp$
$-\frac{c}{2}p + b - c - bp + cp = 0$
$\frac{c}{2}p + b - c - bp = 0$
$(\frac{c}{2} - b)p = c - b$
$p = \frac{c-b}{\frac{c}{2}-b}$
$p = \frac{b-c}{b-\frac{c}{2}}$
$p = \frac{2b-2c}{2b-c}$

$1-p = 1 - \frac{2b-2c}{2b-c}$
$1-p = \frac{2b-c-2b+2c}{2b-c}$
$1-p = \frac{c}{2b-c}$

We will verify that the game has a unique symmetric Nash equilibrium (S,S) where S = $(\frac{2b-2c}{2b-c}, \frac{c}{2b-c})$, and that S is also the unique ESS of the game. To confirm that S is an ESS, we need that for all $S' \neq S, u(S,S) = u(S',S)$ and $u(S,S') > u(S',S')$. The equality condition $u(S,S) = u(S',S)$ is true of any mixed strategy equilibrium with full support, so follows directly. To demonstrate that the inequality holds, it is sufficient to find the S' - equivalently, the probability C - that minimizes f(S') = u(S,S') - u(S',S') as shown below. Let p be our $\frac{2b-2c}{2b-c}$ strategy, S, for playing C. Let q be some other strategy, S', for playing C.
$f(S') = u(S,S') - u(S',S')$
$f(S') = ((b-\frac{c}{2})pq + (b-c)p(1-q) + b(1-p)q + 0(1-p)(1-q)) - ((b-\frac{c}{2})qq + (b-c)q(1-q) + b(1-q)q + 0(1-q)(1-q))$
$f(S') = ((b-\frac{c}{2})pq + (b-c)p(1-q) + b(1-p)q) - ((b-\frac{c}{2})qq + (b-c)q(1-q) + b(1-q)q)$
$f(S') = ((b-\frac{c}{2})pq + (b-c)p - (b-c)pq + bq - bpq) - ((b-\frac{c}{2})qq + (b-c)q - (b-c)qq + bq - bqq$
$f(S') = ((\frac{c}{2}-b)pq + (b-c)p + bq) - ((\frac{c}{2}-b)qq + (2b-c)q)$
$f(S') = (\frac{c}{2}-b)pq - (\frac{c}{2}-b)qq + (b-c)p + (c-b)q$
Substitute $p = \frac{2b-2c}{2b-c}$, $1-p = \frac{c}{2b-c}$
$f(S') = (\frac{c}{2}-b)(\frac{2b-2c}{2b-c})q - (\frac{c}{2}-b)qq + (b-c)(\frac{2b-2c}{2b-c}) + (c-b)q$
$f(S') = (\frac{(2b-2c)(\frac{c}{2}-b)}{2b-c})q - (\frac{c}{2}-b)qq + (\frac{(2b-2c)(b-c)}{2b-c}) + (c-b)q$
$f(S') = (\frac{bc-c^2-2b^2+2bc}{2b-c})q - (\frac{c}{2}-b)qq + (\frac{2b^2-2bc-2bc+2c^2}{2b-c}) + (c-b)q$
$f(S') = (\frac{-2b^2-c^2+3bc}{2b-c})q - (\frac{c}{2}-b)qq + (\frac{2b^2+2c^2-4bc}{2b-c}) + (c-b)q$
$f(S') = -(\frac{c}{2}-b)qq + ((c-b) + \frac{-2b^2-c^2+3bc}{2b-c})q + (\frac{2b^2+2c^2-4bc}{2b-c})$
$f(S') = -(\frac{c}{2}-b)qq + (\frac{2bc-c^2-2b^2+bc}{2b-c} + \frac{-2b^2-c^2+3bc}{2b-c})q + (\frac{2b^2+2c^2-4bc}{2b-c})$
$f(S') = -(\frac{c}{2}-b)qq + (\frac{2bc-c^2-2b^2+bc-2b^2-c^2+3bc}{2b-c})q + (\frac{2b^2+2c^2-4bc}{2b-c})$
$f(S') = -(\frac{c}{2}-b)qq + (\frac{-4b^2-2c^2+6bc}{2b-c})q + (\frac{2b^2+2c^2-4bc}{2b-c})$
$f(S') = (b-\frac{c}{2})qq + (\frac{-4b^2-2c^2+6bc}{2b-c})q + (\frac{2b^2+2c^2-4bc}{2b-c})$
Solve for q that minimizes f(S') by setting derivative of f(S') equal to 0:
$df(S')/dq = 2(b-\frac{c}{2})q + \frac{-4b^2-2c^2+6bc}{2b-c} = 0$
$2(b-\frac{c}{2})q + \frac{-4b^2-2c^2+6bc}{2b-c} = 0$
$(2b-c)q = \frac{4b^2+2c^2-6bc}{2b-c}$
$q = \frac{4b^2+2c^2-6bc}{(2b-c)^2}$
$q = \frac{(2b-2c)(2b-c)}{(2b-c)^2}$
$q = \frac{2b-2c}{2b-c}$
So to minimize f(S'), we see $p = q = \frac{2b-2c}{2b-c}$ and $1-p = 1-q = \frac{c}{2b-c}$. Hence there is a unique maximum when S'=S (p=q) proving our result that for all $S' \neq S, u(S,S') > u(S',S')$. Hence our mixed strategy Nash equilibrium must be a unique ESS. (Note: Proof explanation comes from Section 7.7.2, p.218 in the textbook [10].)